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RADIATION OF INTERNAL WAVES DURING VERTICAL MOTION OF A BODY THROUGH A NONUNIFORM LIQUID

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Energy losses to radiation of internal waves during the vertical motion of a point dipole in two-dimensional and three-dimensional cases are computed.

During the motion of bodies in a liquid with nonuniform density in the field of gravity, in addition to sound waves, internal gravitational waves are also excited in the liquid, and the body due to the loss of wave momentum experiences an additional wave resistance. A simpler problem concerning the motion of singular sources, in some sense approximately equivalent to the bodies, is often examined within the framework of the linear description of the wave field [1-4]. Such substitutions are well known and have a precise meaning in the theory of a uniform ideal liquid. It is assumed that they can also be used in the case of a weakly nonuniform liquid. In what follows, within the framework of a similar approach, we compute the total energy losses due to the formation of waves during vertical motion of bodies with subsonic speeds.

Neglecting dissipation processes in the liquid, its motion as excited by a source with mass ρm can be described with the help of the equations describing the balance of forces and mass and the condition of adiabaticity. If in the absence of the source, the liquid is stationary and the pressure $p_0(z)$ and density $\rho_0(z)$ depend only on the vertical coordinate z, then for small perturbations the basic equations can be written in the following linearized form [5, 6]:

$$\rho_0\hat{D}\mathbf{v}+\nabla p=\rho\mathbf{g},\ \hat{D}\rho+\rho_0 wH^{-1}+\rho_0\nabla\mathbf{v}=\rho_0 m,$$
$$\hat{D}p+\rho_0 wg=c_0^2(\hat{D}\rho+\rho_0 wH^{-1}),\ \ H^{-1}\equiv\frac{d\ln\rho_0}{dz}, \quad (1)$$

where $\hat{D}$ denotes the operator for differentiation with respect to time.

From this system of equations, it is easy to obtain separate equations for perturbations of pressure, density, and velocity. The pressure equation in the case of a liquid with constant coefficients H and $N^2 \equiv gH^{-1} - g^2c_0^{-2}$ can be put into the form

$$\hat{L}p^0=-\left(\frac{\partial^2}{\partial t^2}+N^2\right)\frac{\partial}{\partial t}m^0,$$

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$$\hat{L} \equiv -\frac{1}{c_0^2}\hat{D}^4 + \left(\nabla^2 - \frac{1}{4H^2}\right)\hat{D}^2 + N^2\nabla_h^2, \quad p^0 \equiv \frac{p}{\sqrt{\rho_0}}, \quad m^0 \equiv m\sqrt{\rho_0}. \tag{2}$$

Here, and in what follows, a vector with index h denotes a vector with a zero vertical component (a horizontal vector).

The formula for the pressure can be rewritten in integral form with the help of the retarded Green's function $G^{ret}(\mathbf{r}, t)$. In this case, it is convenient to use a Fourier expansion in terms of plane waves $E \equiv \exp(i\mathbf{kr} - i\omega t)$:

$$m^0 = \frac{1}{(2\pi)^{n+1}} \int d^n k d\omega \mu(\mathbf{k}, \omega) E,$$

$$p^0 = \frac{i}{(2\pi)^{n+1}} \int d^n k d\omega\, \omega (N^2 - \omega^2) G^{ret}(\mathbf{k}, \omega) \mu(\mathbf{k}, \omega) E, \tag{3}$$

where n is the dimensionality of the space, equal in what follows to two or three in the planar and three-dimensional cases, while $d^n k$ is the volume element of the space of wave vectors $\mathbf{k}$.

If a mass source of constant intensity moves uniformly and with rectilinear motion, then $m^0 = \mu_0 f(\mathbf{r} - \mathbf{v}_0 t)$, $\mu(\mathbf{k}, \omega) = 2\pi\mu_0 f(\mathbf{k})\delta(\omega - \mathbf{kv}_0)$, and in the approximation of an ideal liquid, all energy losses by the source are due to the radiation of waves.* Formally, this expresses the equation for conservation of energy in integral form, which follows from the system of equation (1):

$$\oint d\sigma p\mathbf{v} = \int d^n r pm \equiv W,$$

according to which the flow of wave energy through a closed surface equals the energy losses by the source per unit time. In what follows, the energy transmitted to the waves will be estimated according to a more easily computed loss quantity W. Substituting expansion (3) into the expression for W, after simple transformations for the case of uniform rectilinear motion of the source, we obtain the following general formula:

$$W = \frac{i\mu_0^2}{(2\pi)^n} \int d^n k d\omega\, \omega (N^2 - \omega^2) G^{ret}(\mathbf{k}, \omega) |f(\mathbf{k})|^2 \delta(\omega - \mathbf{kv}_0).$$

For the Fourier transform of the retarded Green's function, determined as a solution to the equation

$$\hat{L} G^{ret}(\mathbf{r}, t) = \delta(\mathbf{r})\delta(t), \quad G^{ret}(\mathbf{r}, t)|_{t<0} = 0,$$

we have the formula

$$G^{ret}(\mathbf{k}, \omega) = \left[-\frac{1}{c_0^2}(\omega + i\varepsilon)^4 + \left(\mathbf{k}^2 + \frac{1}{4H^2}\right)(\omega + i\varepsilon)^2 - N^2 k_h^2\right]^{-1},$$

in which the addition of an imaginary infinitely small term $i\varepsilon$ to the frequency corresponds to the property of causality in the Green's function. An important property of the latter expression is its evenness with respect to the components of the wave vector. Keeping this in mind, it is easy to show that only that part of the function $G^{ret}(\mathbf{k}, \omega)$ that is uneven with respect to frequency gives a nonzero contribution to W, i.e., $\operatorname{Im} G^{ret}(\mathbf{k}, \omega)$, and, in this manner, the general formula for W can be rewritten in the following form:

$$W = -\frac{\mu_0^2}{(2\pi)^n} \int d^n k d\omega\, \omega (N^2 - \omega^2) |f(\mathbf{k})|^2 \delta(\omega - \mathbf{kv}_0) \operatorname{Im} G^{ret}(\mathbf{k}, \omega). \tag{4}$$

The expression for $\operatorname{Im} G^{ret}(\mathbf{k}, \omega)$ is found from $G^{ret}(\mathbf{k}, \omega)$ with the help of a well-known formula from the theory of generalized functions $(x + i\varepsilon)^{-1} = Px^{-1} - i\pi\delta(x)$ and we have the form

*The source can experience retardation due to these losses and then, the indicated assumption concerning the uniformity of motion can be satisfied only approximately depending on the relative smallness of the losses. On the other hand, the losses can be compensated by the work of external forces.

$$\operatorname{Im} G^{\mathrm{ret}} = -\pi \operatorname{sgn}\left[\omega\left(N^2 k_h^2 - \frac{\omega^2}{c_0^2}\right)\right] \delta\left(-\frac{\omega^4}{c_0^2} + \omega^2 \mathbf{k}^2 + \frac{\omega^2}{4H^2} - N^2 k_h^2\right).$$

In the three-dimensional case, we consider an axisymmetric body, which is modeled as a one-dimensional distributed source with $f(\mathbf{r}) = \nu(z)\delta(x)\delta(y)$ and moves in a vertical direction along the axis of symmetry ($v_{0x} = v_{0y} = 0$). Then, with the use of cylindrical coordinates k_z, θ, $k_h \equiv (k_x^2 + k_y^2)^{1/2}$, integration with respect to these variables is easily carried out (with respect to angle, due to the cylindrical symmetry of the problem and the function $\operatorname{Im} G^{\mathrm{ret}}(\mathbf{k},\omega)$, and with respect to k_z, k_h due to the product of two δ-functions in the integrand). For simplicity, we will limit ourselves to the analysis of the motion of a source with speeds that are less than the speed of sound. Then, there is no radiation of sound and all losses will be due to the radiation of internal waves. The magnitude of the losses W in any case for $v_0^2 < 2c_0^2$ in general turns out not to depend on the compressibility of the fluid:

$$W = \frac{\mu_0^2 v_0}{4\pi} \int\limits_0^{N/v_0} dk_z k_z |\nu(k_z)|^2. \tag{5}$$

As is well known [7], the motion of a sphere in the approximation of a uniform ideal liquid corresponds to the motion of a point dipole (doublet). For the doublet, $\mu_0 \nu(z) = -d\partial\delta(z)/\partial z$, and in the case of the stratified liquid considered here, the formula for the energy losses due to radiation of internal waves follows from (5)

$$W = \frac{d^2 N^4}{16\pi v_0^3}. \tag{6}$$

In the case of a dipole, consisting of a point source and a sink equal in intensity, separated by a finite distance λ_0 from one another, the integration in (5) is also easily carried out:

$$W = \frac{\mu_0^2 N^2}{4\pi v_0}\left[1 - \frac{2}{\lambda^2}(\lambda \sin\lambda + \cos\lambda - 1)\right], \quad \lambda \equiv \frac{N\lambda_0}{v_0}.$$

In the limit $\lambda_0 \to 0$, $\mu_0 \to \infty$, $\mu_0\lambda_0 = d = \mathrm{const}$, formula (6) follows from here.

The integration is also easily carried out for a smeared out source of the type $\nu(z) = l^{-1}\exp(-z^2 l^{-2})$, where $\nu(k_z) = \sqrt{\pi}\exp(-k_z^2 l^2)$,

$$W = \frac{\mu_0^2 v_0}{16 l^2}\left[1 - \exp\left(-\frac{2l^2N^2}{v_0^2}\right)\right].$$

For large Froude numbers ($v_0/Nl \gg 1$), the dependence on the smearing scale disappears and there is an inverse proportionality with the speed ($W \approx 0.125\mu_0^2 N^2 v_0^{-1}$), while for small Froude numbers $2W \approx 0.125\mu_0^2 v_0 l^{-2}$. From here, the nonuniformity in the limiting transition to the stationary point source is evident. If at first the limit $v_0 \to 0$ is taken, then we obtain $W = 0$, while if the limit $l \to 0$ is taken, and then $v_0 \to 0$, we arrive at an infinitely large value.

In the two-dimensional case, the flow around the cross section of a circular cylinder by a uniform ideal liquid corresponds to the flow around a flat doublet. In the subsonic regime for motion of a doublet in a nonuniform liquid, the expression for the energy losses (4) can be reduced to elliptic integrals. The final answer becomes expecially simple with the condition $v_0 \ll c_0$, NH:

$$W = \frac{d^2 N^3}{6\pi v_0^2}. \tag{7}$$

Simple formulas of type (6) and (7) can be obtained with a precision up to numerical factors using dimensional analysis. For relatively slow motion of a point dipole ($v_0 \ll c_0$), the determining parameters are the characteristics of the doublet d, v_0, and the characteristics of the medium N, H. Within the framework of the linearized description, the radiated energy must be proportional to the square of the source amplitude, i.e., d^2. Therefore, with the help of dimensional considerations

$$W = \frac{d^2 N^{n+1}}{v_0^n} f_n\left(\frac{v_0}{NH}\right).$$

Here $f_n(x)$ is a universal dimensionless function which in the limit $v_0/NH \to 0$ becomes a universal constant C_n, and then formulas of type (6) and (7) are obtained. The fact that the constants C_n then turn out to be finite is not trivial. Indeed, for a point source in the two-dimensional case, in the limit $v_0/NH \to 0$, the quantity W becomes infinitely large. In the three-dimensional case, the losses for a doublet also become infinitely large, if the doublet moves horizontally.

If we use the usual relations relating the dipole moment d to the radius of the body around which the flow occurs $d^2 = (2\pi v_0 r_0^n)^2$, then the formula for the losses of a vertically moving doublet take the form

$$\frac{W}{\pi r_0^4 N^3} = \begin{cases} \dfrac{2}{3}, & n = 2, \\ \dfrac{r_0^2 N}{4 v_0}, & n = 3, \end{cases}$$

so that in the two-dimensional case the function of velocity disappears, while in the three-dimensional case it appears as a decreasing function.

Now, for $v_0 \ll c_0$, NH, the determining parameters for a doublet will be r_0, v_0, and N, so that

$$W = \frac{r_0^{2n} N^{n+1}}{v_0^{n-2}} F_n\left(\frac{v_0}{r_0 N}\right).$$

If for large Froude numbers v_0/r_0N the quantities $F_n(\infty)$ become finite numerical coefficients, then we arrive at the formula of the type obtained previously. However, now this assumption is less justified, although it has a clear physical meaning (in this limit W turns out to be proportional to the square of the volume of the body). Finally, it should be emphasized that the similarity between the problems of the motion of a body and that of a doublet is expected exactly in this limiting case of large Froude numbers.

NOTATION

$\rho_0(z)$, $p_0(z)$, density and pressure of the ground state; z, vertical coordinate; $\mathbf{v}$, p, ρ, perturbed velocity, pressure, and density; $H \equiv (d \ln \rho_0/dz)^{-1}$, characteristic length scale for stratification; $N = (gH^{-1} - g^2 c_0^{-2})^{1/2}$, Weisel–Brent frequency; g, acceleration of gravity; c_0, speed of sound; w, vertical component of the perturbed velocity; ∇, vector operator; $\mathbf{k}$, wave vector; ω, frequency; $d\sigma$, vector surface element; W, magnitude of the energy losses; $\delta(t)$, $\delta(\mathbf{r}) \equiv \delta(x)\delta(y)\delta(z)$, Dirac functions; v_0, velocity of motion of the source of perturbations; d, dipole moment of the doublet; λ_0, l, length dimension parameters; μ_0, intensity of the source.

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